1. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (a) Prove that if $\mu(X) < \infty$ and if $1 \le p < q < \infty$, then $L^q(\mu) \subseteq L^p(\mu)$.
   (b) Is the statement in (a) true if $\mu(X) = \infty$? If yes, prove it. If no, give a counterexample.

2. Let μ and ν be finite measures on a measurable space $(X, \mathcal{M})$ and suppose that

$$\nu(E) = \int_E f \, d\mu$$

for every $E \in \mathcal{M}$, where f is some function in $L^1(\mu)$. Prove that

$$\int_X g \, d\nu = \int_X gf \, d\mu$$

for all $g \in L^1(\nu)$.

3. (a) What is a simple function? What is an integrable function? How is $\int_X f \, d\mu$ defined? Define it first for a simple function, then for a nonnegative measurable function, and then finally for an integrable function.
   (b) Prove that $\int_E f \, d\mu = 0$ if $\mu(E) = 0$ and f is measurable.
   (c) What is the connection between f being integrable and $|f|$ being integrable? Prove it. Also, find a function $f : [a, b] \to \mathbb{R}$ for which $|f|$ is integrable, but f is not Borel measurable.
   (d) State and prove the vanishing principle.

4. (a) What is an $\mathcal{A}$-measurable function $f : X \to [-\infty, +\infty]$? Mention some equivalent conditions.
   (b) What is a Borel measurable function?
   (c) Show how to construct a set which is Lebesgue measurable, but which is not a Borel set.
   (d) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is increasing, then f is Borel measurable.

5. Let $\phi \in C_0^\infty(\mathbb{R})$ be such that ϕ is even, $\|\phi\|_{L^1} = 1$, $\phi(x) \ge 0$, and $\phi(x) = 0$ for $|x| > 1$. Define $\phi_t(x) = t^{-1}\phi(x/t)$.
   (a) Prove that if $f \in C_0(\mathbb{R})$, then $f * \phi_t \to f$ as $t \to 0$ pointwise on $\mathbb{R}$ and that $f * \phi_t \to f$ as $t \to 0$ in the L^p norm for each $1 \le p < \infty$.
   (b) Use part (a) to show that the same result extends to the case where $f \in L^p(\mathbb{R})$. That is, if $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then $f * \phi_t \to f$ as $t \to 0$ in the L^p norm.

6. Suppose that a series $\sum_{n=1}^\infty f_n$ converges absolutely in $L^1(\mathbb{R})$, i.e., $\sum_{n=1}^\infty \|f_n\|_{L^1} < \infty$. Prove that:
   (a) the series $f(x) = \sum_{n=1}^\infty f_n(x)$ converges for a.e. $x \in \mathbb{R}$;
   (b) $f \in L^1(\mathbb{R})$;
   (c) $\int_{\mathbb{R}} \sum_{n=1}^\infty f_n = \sum_{n=1}^\infty \int_{\mathbb{R}} f_n$.

7. Do the following three parts.
   (a) Give precise statements (e.g., as stated in class or in Folland's book) of Tonelli's theorem and Fubini's theorem for general measure spaces.
   (b) Let λ denote Lebesgue measure on $\mathbb{R}$, and let $\lambda^2 := \lambda \times \lambda$ denote the product measure on $\mathbb{R}^2$. Prove that if $f, g \in L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}^2} f(x)g(t-x) \, d\lambda^2(x,t) = \int_{\mathbb{R}^2} g(x)f(t-x) \, d\lambda^2(x,t).$$

   Justify any steps in your proof, and if you use any theorems explain why the hypotheses of those theorems are satisfied.

(c) Let $I^2 := [0,1] \times [0,1]$. Prove that the function $f : I^2 \to \mathbb{R}$ defined by

$$f(x,y) := \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is not integrable. (Hint: It may help to realize that $\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) = f(x,y)$.)

8. Prove that

$$\lim_{n\to\infty} \int_0^\infty \frac{x^n \cos(x/n)}{(1+x^n)e^x}\, dx$$

exists and compute it.

9. (On translation) For a real-valued function f on $\mathbb{R}$, define the translate f^t by $f^t(x) = f(x-t)$.
   (a) Suppose f is continuous on $\mathbb{R}$ and has compact support. Prove that $\|f^t - f\|_{L^\infty(\mathbb{R})} \to 0$ as $t \to 0$.
   (b) Show that if $f \in L^1(\mathbb{R})$, then $\|f^t - f\|_{L^1(\mathbb{R})} \to 0$ as $t \to 0$.

10. (a) State Lebesgue's dominated convergence theorem.
    (b) Let μ be counting measure on $\mathbb{N}$. We can identify a function $f : \mathbb{N} \to \mathbb{R}$ with a sequence $(a_n)_{n\in\mathbb{N}}$. Prove that if f is integrable then $\sum_{k=1}^\infty |a_k|$ converges in the sense of Calculus 2, and the sum equals $\int_{\mathbb{N}} |f|\, d\mu$. Show also that in this case $\int_X f\, d\mu = \sum_{k=1}^\infty a_k$.
    (c) Show that a Riemann integrable function $f : [a,b] \to \mathbb{R}$ is Lebesgue integrable, and that the two integrals of f coincide.
    (d) Complete the sentence: a bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if $f \ldots$.

11. (a) State LDCT.
    (b) What is a 'Banach space'?
    (c) For a measure space (X,μ), show that the simple functions are dense in $L^1(X,\mu)$.
    (d) Show that $L^\infty([0,1])$ is not separable.

12. Prove the following statements or disprove by providing a counterexample:
    (a) If a set $E \subset \mathbb{R}^n$ has nonempty interior, then $\lambda^*(E) > 0$. (Note: λ^* is the outer measure.)
    (b) If a set $E \subset \mathbb{R}^n$ is such that $\lambda^*(E) > 0$, then E has nonempty interior.
    (c) Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is measurable.
    (d) If $f : \mathbb{R} \to \mathbb{R}$ is continuous at all but one point, then there exists a continuous function g such that $f = g$ a.e.
    (e) If $A \subset \mathbb{R}$ is an uncountable Lebesgue-measurable set, the its Lebesgue measure is larger than 0.
    (f) If $\{f_k\}$ is a sequence of non-negative and measurable functions on $[0,1]$ such that $f_k \to 0$ a.e., then $\int_{[0,1]} f_k \to 0$.
    (g) If $\{f_k\}$ is a sequence of non-negative and measurable functions on $[0,1]$ such that $f_k \leq M$ for all k and $f_k \to 0$ a.e., then $\int_{[0,1]} f_k \to 0$.
    (h) If $\{f_k\}$ is a sequence of non-negative and measurable functions on $[0,1]$ such that $f_k \to 0$ in measure, then $f_k \to 0$ a.e.

13. Let $1 \leq p < \infty$ and suppose that $f_n, f \in L^p(\mathbb{R}^n)$ satisfy $f_n \to f$ a.e. Prove that

$$\|f_n - f\|_{L^p} \to 0 \iff \|f_n\|_{L^p} \to \|f\|_{L^p}.$$

14. Let $\mathcal{M}$ be the collection of all subsets E of $\mathbb{R}$ such that either E or its complement is at most countable. Prove that $\mathcal{M}$ is a σ-algebra.

15. Let $f \in L^1((0,1))$. Define g on $(0,1)$ by

$$g(x) = \int_x^1 \frac{f(t)}{t}\, dt.$$

Prove that $g \in L^1((0,1))$.

16. (Absolute continuity)
    (a) A function $f : [0,1] \to \mathbb{R}$ is said to be *Lipschitz* if there exists $L > 0$ such that $|f(x)-f(y)| \leq L|x-y|$ for all $x, y \in [0,1]$. Prove that if $f : [0,1] \to \mathbb{R}$ is Lipschitz, then f is absolutely continuous.
    (b) Suppose $f : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and absolutely continuous on $[\varepsilon, 1]$ for every $\varepsilon \in (0,1)$. Show that f need not be absolutely continuous on $[0,1]$ by giving a counterexample. Hint: Consider functions of the form $x^\alpha \cos(1/x^b)$.

17. Let λ denote Lebesgue measure on $[0,1]$. Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos(2\pi kx) + x^k}{k^2}$$

is an integrable function and that

$$\int_0^1 f(x)\, d\lambda(x) = \sum_{k=1}^{\infty} \frac{1}{(k+1)k^2}.$$

18. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $L^1([0,1])$ with $\|f_n\|_1 \leq K$ for each $n \in \mathbb{N}$. Prove that if $g : [0,1] \to \mathbb{C}$ and $f_n \to g$ almost everywhere, then $g \in L^1([0,1])$ and $\|g\|_1 \leq K$.

19. (a) What is Lebesgue outer measure λ^* on $\mathbb{R}^n$? What does it mean for a subset A of $\mathbb{R}^n$ to be Lebesgue measurable?
    (b) Spell out all the relationships between the Lebesgue σ-algebra and the Borel σ-algebra. You don't need to prove anything.
    (c) Explain the Caratheodory construction/theorem.
    (d) What does it mean to say Lebesgue measure is a complete measure? Explain briefly how we know that Lebesgue measure is complete.
    (e) Show that there exists a Lebesgue measurable subset of $\mathbb{R}$ which is not a Borel set. (You don't need to prove anything about the ternary function, just use it.)

20. (a) Let h and g be integrable functions on X and Y respectively, and let $f(x,y) = h(x)g(y)$. Show that f is integrable and $\int f\, d(\mu \times \nu) = (\int_X h\, d\mu)(\int_Y g\, d\nu)$. State and prove the corresponding theorem if h, g are measurable and nonnegative.
    (b) Discuss some of the various types of convergence $f_n \to f$, and how they are related. You don't need to prove anything, but you might give some examples.
    (c) Define the convolution $f * g$ of two functions in $L^1(\mathbb{R})$. Show that the function inside the integral in the definition of $f * g$ is measurable. Show also that $f * g = g * f$.
    (d) Show that the Fourier transform of $f * g$ is a product of the Fourier transform of f and the Fourier transform of g.

21. (a) What is a complex measure ν on $(X, \mathcal{A})$? (If you never met complex measures, you may switch them with finite signed measures here and in questions below.)
    (b) Define the variation $|\nu|$. Also, complete the sentence: "The variation $|\nu|$ is the smallest positive measure such that ...." Define $\|\nu\|$.

(c) Prove that $|\nu+\sigma| \leq |\nu|+|\sigma|$ and $|c\nu| = |c||\nu|$ for complex measures ν, σ on $(X, \mathcal{A})$, and deduce that the space $M(X)$ of such complex measures is a normed space.

(d) Prove that $L^2(X, \mu)$ is a Hilbert space.

22. (a) Consider a compact Hausdorff space K, and let $C(K)$ be the continuous scalar functions on K. Define the supremum norm on $C(K)$, show that it is a norm, and that with this norm $C(K)$ is a normed space. If you are lost, do this in the case $K = [0, 1]$.

(b) With notation in (a), explain carefully the relationship between the measures on K and linear functionals on $C(K)$. Consider two cases, the positive (or nonnegative) case and the general case.

(c) What can you say about a surjective linear continuous one-to-one function between Banach spaces? Also state the principle of uniform boundedness.

(d) Define the terms "dense" and "separable." Give a simple test for when a normed space is not separable, and using this show that $L^\infty([0, 1])$ is not separable.

23. (a) Define $L^p(X, \mu)$ for $1 \leq p \leq \infty$.

(b) Prove in full detail that $L^1(X, \mu)$ is a normed space and a Banach space.

(c) Prove that $L^2(X, \mu)$ is a Hilbert space giving all the details. You can assume that it is a Banach space.

(d) If $g \in L^1(X, \mu)$, with $g \geq 0$, and $\nu(E) = \int_E g\, d\mu$, show that ν is a measure, and that $\int_X f\, d\nu = \int_X fg\, d\mu$ if either f is measurable and $f \geq 0$ or $f \in L^\infty(X, \mu)$.

(e) Complete: The 'density' of a finite measure on $\mathbb{R}^n$ equals ________________ derivative a.e. That is, for a.e. $\vec{x}$, if measurable sets E_n 'shrink nicely to' $\vec{x}$, then $\lim_{k\to\infty} \frac{\nu(E_k)}{\lambda(E_k)} =$ _________.

24. (a) Show that if f is a continuous scalar valued function on $[a, b]$, and if $f = 0$ a.e., then $f = 0$ everywhere.

(b) What is the connection between the Riemann and the Lebesgue integral?

(c) Show that the Lebesgue integral is translation invariant.

(d) Show that for $f, g : [a, b] \to \mathbb{R}$, if $f = g$ a.e., and g is continuous, then it does not necessarily follow that f is continuous a.e.

25. (a) What does it mean that measurable sets E_n 'shrink nicely' to x?

(b) Complete the sentence: "If $f \in L^1(\mathbb{R}^k)$ and if x is a Lebesgue point of f and E_n shrink nicely to x, then ...."

(c) State and prove the 'first fundamental theorem of calculus' involving NAC.

(d) Prove that a function on $\mathbb{R}$ is Lipschitz iff it is in AC and its derivative is in L^∞.

26. (a) For $1 < p < \infty$ set $q = p'$, and define a map $\Phi : L^q(X, \mu) \to (L^p(X, \mu))^*$, showing that it is a well-defined linear isometry. State a theorem from class about when Φ is also surjective.

(b) Define the terms: AC (absolute continuity), NBV.

(c) Explain the fundamental correspondence between measures on $\mathbb{R}$ and NBV, and between $L^1(\mathbb{R})$ and AC.

(d) Show that every $f \in AC[a, b]$ is of bounded variation.

27. Let $E \subset \mathbb{R}^n$ be a measurable set and f an integrable function on E.

(a) Set $E_m = \{|f| < m\}$ and show that $f\chi_{E_m} \to f$ in L^1-norm as $m \to \infty$.

(b) Given $\epsilon > 0$, show that there exists a constant $\delta > 0$ such that for every measurable set $A \subset E$, we have

$$\lambda(A) < \epsilon \implies \int_A |f| < \epsilon.$$

(c) Show that, given $\epsilon > 0$, there exists a measurable set $A \subset E$ such that f is bounded on A and $\int_{E \setminus A} |f| < \epsilon$.

28. Suppose that $f : [0,1]^2 \to \mathbb{R}$ satisfies the following conditions:
   i. For each fixed $x \in [0,1]$, $f(x,y)$ is an integrable function of y;
   ii. $\frac{\partial f}{\partial x}(x,y)$ exists at all points and is bounded on $[0,1]^2$.

   Prove that $\frac{\partial f}{\partial x}(x,y)$ is a measurable function of y for each $x \in [0,1]$ and that

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy.$$

29. Fix $1 \leq p < \infty$ and suppose that:
   i. $f_m \to f$ in measure;
   ii. for each $\epsilon > 0$, there exists a $\delta > 0$ such that for every measurable set $E \subset \mathbb{R}^n$ satisfying $\lambda(E) < \delta$, we have $\int_E |f_m|^p < \epsilon$ for every m;
   iii. for each $\epsilon > 0$, there exists a measurable set $E \subset \mathbb{R}^n$ such that $\lambda(E) < \infty$ and $\int_{E^c} |f_m|^p < \epsilon$ for every m.

   Prove that $f_m \to f$ in $L^p(\mathbb{R}^n)$.

30. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (a) Let $f : X \to [0, +\infty]$ be measurable and suppose that $\int_X f\,d\mu = 0$. Prove that $f = 0$ a.e.
   (b) Let $f \in L^1(X, \mu)$. Show that if $\int_A f\,d\mu = 0$ for every $A \in \mathcal{M}$, then $f = 0$ a.e.

31. Let $k \in L^1(\mathbb{R})$ be such that $\int_{\mathbb{R}} k(x)\,dx = 1$ and, for each $n \in \mathbb{N}$, set $k_n(x) = nk(nx)$. Prove that for every $f \in L^1(\mathbb{R})$,

$$\lim_{n\to\infty} \|f * k_n - f\|_{L^1} = 0.$$

   Recall that the convolution is defined by $f * k(x) = \int_{\mathbb{R}} f(x-y)k(y)\,dy$.

32. (On a property of Lebesgue integrable functions)

   Let $f \in L^1(\mathbb{R})$.
   (a) Fix $\alpha > 0$. For $n \in \mathbb{N}$, define f_n by $f_n(x) = f(nx)/n^\alpha$. Show that

$$\|f_n\|_1 = \int_{\mathbb{R}} \frac{|f(nx)|}{n^\alpha}\,dx = \int_{\mathbb{R}} \frac{|f(z)|}{n^{1+\alpha}}\,dz = \frac{\|f\|_1}{n^{1+\alpha}}.$$

   (b) Use (a) to show that $f_n(x) \to 0$ as $n \to \infty$ for m-a.e. $x \in \mathbb{R}$ (where m is Lebesgue measure on $\mathbb{R}$).

33. Let $F \subset \mathbb{R}$ be a closed subset of positive measure. For $x \in \mathbb{R}$, define the distance from x to F by

$$d(x,F) = \inf_{z\in F} d(x,z).$$

   Prove that for Lebesgue almost every $y \in F$, we have

$$\lim_{x\to y} \frac{d(x,F)}{|x-y|} = 0.$$

   Hint: Consider Lebesgue density points of F.

34. (On the closed graph thereom)
   (a) State the closed graph theorem.

(b) Let $\mathfrak{X}$ and $\mathfrak{Y}$ be (complex) Banach spaces. Prove that if $T : \mathfrak{X} \to \mathfrak{Y}$ is a linear map such that $\varphi \circ T \in \mathfrak{X}^*$ for every $\varphi \in \mathfrak{Y}^*$, then T is a bounded linear map. Here $\mathfrak{X}^*$ and $\mathfrak{Y}^*$ denote the dual spaces of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively.

35. (Convolution)

(a) Define the *convolution* $f * g$ of two measurable functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$.

(b) Suppose $f, g \in L^1(\mathbb{R})$. Prove that
$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$
You must prove this special case of the Young inequality yourself; do not simply quote the Young inequality.

(c) Let $\mathcal{F}[\cdot]$ denote the Fourier transform. Suppose that $f, g \in L^1(\mathbb{R})$. Prove that $\mathcal{F}[f * g](\gamma) = \mathcal{F}[f](\gamma)\mathcal{F}[g](\gamma)$ for all $\gamma \in \mathbb{R}$.

(d) Prove that there does not exist $u \in L^1(\mathbb{R})$ such that $f = f * u$ a.e. for every $f \in L^1(\mathbb{R})$. Hint: proceed by contradiction. Assume u exists and let $f \in L^1(\mathbb{R})$ be a function such that $\mathcal{F}[f](\gamma) \neq 0$ for all $\gamma \in \mathbb{R}$ (you do not need to give an explicit example of such an f). Now use
$$\|\mathcal{F}[f - f * u]\|_{L^\infty(\mathbb{R})} \leq \|f - f * u\|_{L^1(\mathbb{R})} = 0$$
to deduce a contradiction.

36. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are real-valued measurable functions such that $\int_X f\, d\mu = \int_X g\, d\mu$. Prove that either $f = g$ μ-almost everywhere or there exists a set $E \in \mathcal{M}$ with $\mu(E) > 0$ and $\int_E f\, d\mu > \int_E g\, d\mu$.

37. (a) If $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces, define the σ-algebra $\mathcal{A} \times \mathcal{B}$. Also explain briefly how the product measure $\mu \times \nu$ is defined.

(b) What are the relationships between $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}^2)$, $\mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R})$, and $\mathcal{L}(\mathbb{R}^2)$? Prove most of your assertions.

(c) State and prove Tonelli's theorem for $\mu \times \nu$.

(d) State and prove Fubini's theorem for $\mu \times \nu$.

38. If $f : X \to [0, \infty]$, define $E = \{(x, y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\}$. This is 'the region under the graph of f'.

(a) Prove that if f is measurable, then E is in $\mathcal{A} \times \mathcal{L}(\mathbb{R})$.

(b) Prove that under the hypothesis of (a), $(\mu \times \lambda)(E) = \int_X f\, d\mu$. Note that some books <u>define</u> the integral by this formula.

39. (a) What is a σ-algebra? What is a measure? What is a measure space?

(b) Prove that if $(X, \mathcal{A}, \mu)$ is a measure space, then μ satisfies the conditions defining an outer measure (except for being defined on all sets).

(c) What is the Borel σ-algebra on a topological space? What is a Borel set?

(d) Show that the set $[0, 1) \cup \mathbb{Q}$ is a Borel set.

40. (a) What is Lebesgue outer measure λ^*, and Lebesgue measure λ on $\mathbb{R}^n$? Outline the construction (you don't need to prove anything).

(b) Show that if A is a Lebesgue measurable set, then $\lambda(A) = \inf\{\lambda(U) : \text{all open sets } U \text{ containing } A\}$, and that this also equals $\sup\{\lambda(K) : \text{compact } K \subset A\}$.

(c) Prove that Lebesgue measure on $\mathbb{R}^n$ is σ-finite.

41. Fix $1 < p < \infty$ and let q satisfy $1/p + 1/q = 1$. Let $(f_n)_{n=1}^\infty$ be a sequence in $L^p([0, 1])$ for which there exists $K > 0$ such that $\|f_n\|_p \leq K$ for every $n \in \mathbb{N}$. Suppose that there exists a Lebesgue measurable function f on $[0, 1]$ such that $f_n(x) \to f(x)$ for m-a.e. $x \in [0, 1]$.

(a) Prove that $f \in L^p([0,1])$ and $\|f\|_p \le K$.

(b) Prove that for every $g \in L^q([0,1])$, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)\,dx = \int_0^1 f(x)g(x)\,dx.$$

(c) Is the statement in part (b) true if $p = 1$ and $q = \infty$? If yes, prove it. If no, give a counterexample.

42. (On modes of convergence.) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(f_n)_{n=1}^{\infty}$ be a sequence of μ-integrable functions and suppose f is μ-integrable as well.

    (a) Prove that if $f_n \to f$ in the $L^1(\mu)$ sense, then $f_n \to f$ in measure.

    (b) If $\mu(X) < \infty$ and if $f_n \to f$ in measure, does it follow that $f_n \to f$ in the $L^1(\mu)$ sense? Either prove this or give a counterexample.

43. (Analysis of a singularity)

    (a) Prove that if $f \in L^p([0,1])$ and if $2 < p < \infty$, then the integral

$$\int_0^1 \frac{|f(x)|}{\sqrt{x}}\,dm(x) \tag{1}$$

    is finite.

    (b) Prove or provide a counterexample: If $f \in L^2([0,1])$, then integral (1) is finite.

44. (An absolute continuity result for the integral)

    Let $f \in L^1(\mathbb{R})$.

    (a) Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f| \cdot \mathbf{1}_{\{x\in\mathbb{R}:|f(x)|>n\}}\,dm = 0.$$

    Here $\mathbf{1}$ denotes the indicator function.

    (b) Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every measurable set $E \subset \mathbb{R}$ satisfying $m(E) < \delta$, we have

$$\int_E |f(x)|\,dm(x) < \varepsilon.$$

45. Suppose $f : [0,1] \to \mathbb{R}$ and $f(0) = 0$. For each of the following three statements about f, indicate whether the statement is TRUE or FALSE. If TRUE, give a proof of the statement. If FALSE, provide a counterexample. (For any counterexample, you can simply describe the function and state its properties; you don't need to prove that it has those properties or go into great detail.)

    (a) If there exists $g \in L^1([0,1])$ such that $f(x) = \int_0^x g(t)\,d\lambda(t)$ for all $x \in [0,1]$, then f is differentiable at almost every $x \in [a,b]$.

    (b) If f is differentiable at almost every $x \in [a,b]$, and $f'(x) = 0$ whenever f is differentiable at $x \in [0,1]$, then $f(1) = 0$.

    (c) If f is absolutely continuous, and $f'(x) = 0$ whenever f is differentiable at $x \in [0,1]$, then $f(1) = 0$.

46. Let $\mathcal{L}$ denote the Lebesgue measurable subsets of $\mathbb{R}$, and let λ denote Lebesgue measure on $(\mathbb{R}, \mathcal{L})$. Define signed measures μ and ν on $(\mathbb{R}, \mathcal{L})$ by

$$\mu(E) := \int_E |x|\,d\lambda(x) \quad \text{and} \quad \nu(E) := \int_{E\cap[-1,\infty)} x\,d\lambda(x).$$

    (a) Prove that $\nu \ll \mu$ and find $\frac{d\nu}{d\mu}$.

    (b) Either prove or disprove that $\mu \ll |\nu|$. (The symbol $|\nu|$ denotes the total variation measure of ν.)

47. (a) Define the words 'Banach space', 'Hilbert space'.
    (b) Prove in full detail that $L^1([a,b])$ is a Banach space.
    (c) State Hölder's inequality.
    (d) Show that the continuous functions on $[a,b]$ are dense in $L^1([a,b])$.

48. Let (X,μ) be a measure space.
    (a) If $\nu(E) = \int_E g\,d\mu$ for all measurable sets E, write down a formula for $\int_X f\,d\nu$. What f, g, ν does your formula hold for?
    (b) State the Radon-Nikodym/Lebesgue decomposition theorem in the case where all measures are finite, and prove the 'uniqueness' part.
    (c) Using (a)-(b) and the fact that $\nu \ll |\nu|$, or otherwise, prove that $|\int_X f\,d\nu| \le \int_X |f|\,d|\nu|$ if f is a bounded measurable function and $\nu \in M(X)$.
    (d) Show that if μ is σ-finite, then $L^1(X,\mu)$ is isometrically isomorphic to the subspace $M_{\mu-a.c.} = \{\nu \in M(X) : \nu \ll \mu\}$ of the Banach space $M(X)$.

49. (a) What is a simple (measurable) function?
    (b) Prove that a product of simple (measurable) functions is a simple (measurable) function.
    (c) Show that if f is a real-valued function on $[a,b]$ that is continuous a.e., then f is Borel measurable and Lebesgue measurable.
    (d) Prove giving all details why if $f, g : X \to [-\infty, \infty]$ are integrable and $f \le g$ a.e., then $\int_X f \le \int_X g$.

50. (a) If $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ are σ-finite measure spaces, define the σ-algebra $\mathcal{A}\times\mathcal{B}$. Also explain briefly how the product measure $\mu\times\nu$ is defined.
    (b) Define the convolution $f * g$ of two functions in $L^1(\mathbb{R})$. Show that the function inside the integral in the definition of $f * g$ is measurable, and that $f * g = g * f$.
    (c) Show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $(f*g)(x)$ exists for all x, and $f * g$ is bounded with $\|f*g\|_\infty \le \|f\|_p \|g\|_q$.
    (d) Prove the formula for the Fourier transform of $f * g$ for $f, g \in L^1(\mathbb{R}^n)$.

51. (a) What is the connection between f being integrable and $|f|$ being integrable?
    (b) What are the relationships between $\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\mathbb{R}^2)$, $\mathcal{L}(\mathbb{R})\times\mathcal{L}(\mathbb{R})$, and $\mathcal{L}(\mathbb{R}^2)$? Prove these.
    (c) State and prove Tonelli's theorem for $\mu\times\nu$.
    (d) Let h and g be integrable functions on X and Y respectively, and let $f(x,y) = h(x)g(y)$. Show that f is integrable and $\int f\,d(\mu\times\nu) = (\int_X h\,d\mu)(\int_Y g\,d\nu)$.

52. (On absolute continuity)

    Let m denote Lebesgue measure on $\mathbb{R}$.
    (a) Let $a < b$ be real numbers. Give the definition of an absolutely continuous function $f : [a,b] \to \mathbb{R}$.
    (b) Suppose $f : [a,b] \to \mathbb{R}$ is absolutely continuous. Prove that if A is a Lebesgue measurable subset of $[a,b]$ with $m(A) = 0$, then $m(f(A)) = 0$.
    (c) If E is a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) = 0$, does it follow that
    $$\{e^x : x \in E\}$$
    has Lebesgue measure zero? Either prove this or give a counterexample.

53. (On the Fourier transform) Let $f : \mathbb{R} \to \mathbb{C}$ be Lebesgue integrable. Recall that the Fourier transform of f is defined by
$$\hat{f}(\gamma) = \int_{\mathbb{R}} f(t)e^{-2\pi i\gamma t}\,dt.$$

(a) Prove that $\hat{f}$ is uniformly continuous on $\mathbb{R}$.

(b) Prove that
$$\lim_{\gamma\to\infty} \hat{f}(\gamma) = 0.$$
Hint: First show this for the characteristic function of an interval of finite length. To complete the proof, make a density argument.

54. Let $(X, \mathcal{M}, \mu)$ be a measure space. Throughout this problem, all functions are real-valued on X and measurable. For each part, either prove the statement or provide a counterexample.
   (a) If $f_n \to f$ in the $L^1(\mu)$ sense, then $f_n \to f$ in measure.
   (b) If $f_n \to f$ in measure and if $\mu(X) < \infty$, then $f_n \to f$ in the $L^1(\mu)$ sense.
   (c) If $f_n \to f$ almost uniformly, then $f_n(x) \to f(x)$ for μ-a.e. $x \in X$. (Recall that $f_n \to f$ almost uniformly if for every $\varepsilon > 0$, there exists $E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus E$.

55. Let m denote Lebesgue measure on $\mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be a measurable function for which there exists $C > 0$ such that $|F(x)| \le C|x|$ for every $x \in \mathbb{R}$. Suppose further that F is differentiable at 0 with $F'(0) = a$. Prove that
$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{nF(x)}{x(1+n^2x^2)}\, dm(x) = \pi a.$$
Hints: Consider the change of variable $u = nx$. You may use the fact that
$$\int_{-\infty}^{\infty} \frac{1}{1+u^2}\, dm(u) = \pi.$$

56. (On weak convergence) Let m denote Lebesgue measure on $[0, 1]$. Let (f_n) be a sequence of functions in $L^2([0, 1])$ that converges weakly to $f \in L^2([0, 1])$, meaning that
$$\lim_{n\to\infty} \int_0^1 f_n g\, dm = \int_0^1 fg\, dm$$
for every $g \in L^2([0, 1])$. Prove that there exists $K > 0$ such that $\|f_n\|_{L^2([0,1])} \le K < \infty$ for every $n \in \mathbb{N}$.
Hint: Uniform boundedness principle.

57. (a) Prove that the dual of L^1 is L^∞, in the case of a finite positive measure.
   (b) What does it mean for a measure to be concentrated on a set?
   (c) State the Hahn and the Jordan decompositions.
   (d) For a finite signed measure ν, show that $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{A},\ F \subset E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{A},\ F \subset E\}$.

58. (a) Define the terms: AC, NAC, Lebesgue-Stieljes measure.
   (b) Complete the sentence: "An NBV function F is in NAC iff its Lebesgue-Stieljes measure..."
   (c) State and prove the 'second fundamental theorem of calculus' involving AC($[a, b]$).
   (d) Show that every $f \in$ AC($[a, b]$) is of bounded variation.

59. (a) Show that every open set in $\mathbb{R}^n$ is a countable disjoint union of 'half open intervals' (that is, Cartesian products of n real intervals of the form $[a, b)$).
   (b) Show that Lebesgue measure is translation invariant.
   (c) What does it mean to say Lebesgue measure is regular? Prove this in $\mathbb{R}$.
   (d) Explain why Borel sets are Lebesgue measurable.

(e) Prove that the sum of two real-valued measurable functions is measurable.

(f) Show that the plane $x + y + z = 0$ has three-dimensional Lebesgue measure equal to 0.

60. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) State the monotone convergence theorem.

(b) Suppose that $\{f_n\}$ is a sequence of nonnegative measurable functions on X that converge pointwise to a function f, and that $f_n \leq f$ for each n. Prove that $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$. (Note that we are not assuming that f is integrable.)

(c) State and prove the Beppo Levi theorem.

(d) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map, state and prove a formula for the Lebesgue measure of $T(E)$ for a Lebesgue measurable set E.

(e) Suppose that E_n are disjoint sets in $\mathcal{A}$ for $n \in \mathbb{N}$, and that f is a NONNEGATIVE measurable function on $X = \cup_n E_n$. Prove that f is measurable on each E_n, and show that $\int_X f = \sum_n \int_{E_n} f$. Deduce that the sum here converges if and only if f is integrable on X.

61. (a) State the Radon-Nikodym/Lebesgue decomposition theorem, in the case where all measures are finite, and prove part of it.

(b) What does it mean to say that a linear map $T : Z \to Y$ between normed spaces is *open*?

(c) State the *open mapping theorem*, the *closed graph theorem*, and a version of the *Hahn-Banach theorem*.

(d) Use the Hahn-Banach theorem to show that the canonical map from a normed space Z into Z^{**} is an isometry.

62. (a) What is a Hilbert space? What does it mean to say that two Hilbert spaces are unitarily isomorphic?

(b) Explain why $L^2([0, 1])$ is unitarily isomorphic to ℓ^2.

(c) Show that for every $f \in L^2([0, 2\pi])$, the Fourier series of f converges to f in L^2-norm.

(d) State Plancherel's theorem.